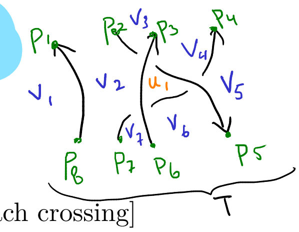
Tangle diagrams

Let T be a tangle diagrams

- P_T := vector space generated by endpoints of T
- V_T := vector space generated by exterior faces of T
- U_T := vector space generated by interior faces of T
- M_T := quadratic form on $U_T \oplus V_T$ by [doing a thing at each crossing]

e.g.



$\Rightarrow$ $P_T = \langle p_i \rangle_{i=1}^{8}$, $V_T = \langle v_i \rangle_{i=1}^{7}$, $U_T = \langle u_1 \rangle$

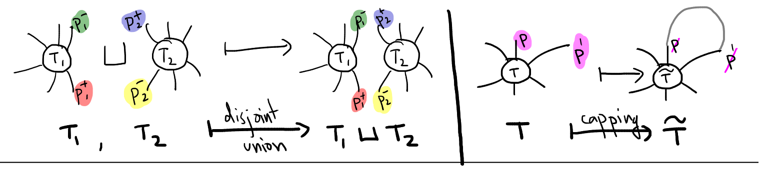
Can embed $\delta : V_T \hookrightarrow P_T$: e.g. $\delta: v_2 \longmapsto p_2 - p_1 + p_8 - p_7$, $v_1 \longmapsto p_1 - p_8$, $v_7 \longmapsto p_7 - p_6$

An **endpoint-preserving** map $\phi : P_{T_1} \to P_{T_2}$ is an isomorphism where

- endpoints go to endpoints of the same type (head or tail)
- If (p, p') are adjacent endpoints, then $(\delta(p), \delta(p'))$ are adjacent endpoints in the same order.

Operations on tangles:

- **Disjoint union**: $T_1, T_2 \mapsto T_1 \sqcup T_2 = T_1 \sqcup^{p_1^+, p_1^-}_{p_2^+, p_2^-} T_2$
- **Capping**: $T \mapsto \widetilde{T} = \widetilde{T}^{p,p'}$



3-tuples

Let P be a vector space. Let $\mathcal{T}(P)$ be 3-tuples $\{(n, W, B)\}$ where:

- $n \in \mathbb{Z}$
- $W \subseteq P$ a subspace
- $B : W \to W^*$ a quadratic form

Operations on 3-tuples:

- Adding: $(n_1, W_1, B_1) + (n_2, W_2, B_2) := (n_1 + n_2, W_1 \oplus W_2, B_1 \oplus B_2 \in \mathcal{T}(P_1 \oplus P_2)$.
- Pullback by $\phi : P_1 \to P_2$: $\phi^*(n_2, W_2, B_2) = (n_2, \phi^{-1}(W_2), \phi^* B_2 \phi)$

A map s from tangle diagrams to 3-tuples

Let s be a map that sends T to $s(T) = (n_T, W_T, B_T) \in \mathcal{T}(P_T)$ where:

- $n_T = \sigma(M|_{U_T})$
- $W_T = \{v \in V_T : M(v)|_{U_T} \in \operatorname{im}(M|_{U_T})\}$
- $B_T : W_T \to W_T^*$ is given by $B_T(v_1)(v_2) := M(v_1 - u_1)(v_2 - u_2)$ where $u_1, u_2 \in U_T$ are such that $M(u_1)|_{U_T} = M(v_1)|_{U_T}$ and $M(u_2)|_{U_T} = M(v_2)|_{U_T}$

Note that:

- B_T does not depend on the choices of u_1, u_2.
- Can decompose $U^* \cong \operatorname{im}(M|_U) \oplus \ker(M|_U)^*$ so can also describe W_T as:
  - $\{v \in V_T : M(v)|_{\ker(M|_U)} = 0\}$ or
  - the image of the projection of $\{w \in U_T \oplus V_T : M(w)|_U = 0\}$ to V_T.

How operations on tangles change s:

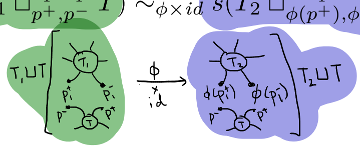
- For disjoint union: $s(T_1 \sqcup T_2) = \mathrm{id}^*(s(T_1) \oplus s(T_2))$
- For capping:
  - If there is $v \in W_T$ that becomes an interior face in $\widetilde{T}$, then $s(\widetilde{T}) = \mathrm{id}^{-1}(n_T + B_T(v)(v), \operatorname{Ann}_{B_T}(v), B_T|_{\operatorname{Ann}_{B_T}(v)})$
  - If there is no such v, then $s(\widetilde{T}) = \mathrm{id}^*(s(T))$

A tangle invariant

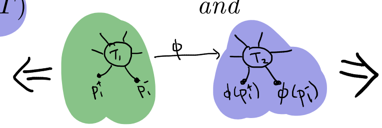
Call $s(T_1)$ and $s(T_2)$ **equivalent** if $s(T_1) = \phi^*(s(T_2))$ for some end-point preserving ϕ. Denote equivalence by $s(T_1) \sim_\phi s(T_2)$. We'll also say that T_1 and T_2 are **s-equivalent** if $s(T_1)$ and $s(T_2)$ are equivalent.

Theorem 0.1. *s plays well with tangle operations. If $s(T_1) \sim_\phi s(T_2)$, then:*

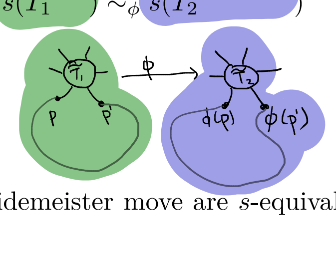
- $s(T_1 \sqcup^{p_1^+, p_1^-}_{p^+, p^-} T) \sim_{\phi \times id} s(T_2 \sqcup^{\phi(p_1^+), \phi(p_1^-)}_{\phi(p^+), \phi(p^-)} T)$ *and* $s(\widetilde{T_1}^{p,p'}) \sim_\phi s(\widetilde{T_2}^{\phi(p),\phi(p')})$



$\Leftarrow$



$\Rightarrow$



Theorem 0.2. *Under equivalence, s is a tangle invariant.*

Proof. By the previous theorem, just need to show that the two sides of each Reidemeister move are s-equivalent. Can do this by using the identity map on the endpoints. $\square$